Robust Synchronization of Lorenz-Rössler Chaotic Systems: Active Sliding Mode and LQG/LTR Design and Comparison

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***Abstract*- In this paper, two robust controllers are designed for the synchronization of Lorenz-Rössler chaotic systems. First an active sliding mode control is presented in which the resulted control parameters not only secure the stability of the error system; they can also be adjusted to obtain the desired rate of error convergence. Second, linear quadratic gaussian/loop transfer recovery (LQG/LTR) controller is proposed which allows the designer to shape the principal gains of the return ratio at either the input or the output of the plant, to achieve robust stability and robust performance specifications. The numerical simulation results show the effectiveness of two proposed methods.**

Keywords- Chaotic Systems; Chaos Synchronization; Active Sliding Mode Control; LQG/LTR Controller; Lorenz System; Rössler System

I. INTRODUCTION

Chaos is an interesting complex dynamical phenomenon which plays an important role in the field of non-linear science. Recently, the traditional trend of analyzing and understanding chaos has evolved into a new phase. Research in this field has been moved to chaos control, synchronization, modeling, and even chaotification [1-7].

Synchronizing the chaotic systems has gained a lot of attention among scientists from a variety of research fields in recent years [1-7]. The chaos synchronization is essential in some applications such as communication and secure communication [4]. The idea of synchronizing two identical chaotic systems was first introduced by Carroll and Pecora [1, 8]. A wide variety of approaches such as active control [9-13] and sliding mode control [15, 16] have been employed for the synchronization of chaotic systems, most of which are designed to synchronize two identical chaotic systems. Bai and Lonngren used active control technique to synchronize two Lorenz systems [9]. The same approach, because of its inherent advantages, has been used by many other researchers to synchronize other chaotic systems [10-13]. The sliding mode is another efficient method for synchronization [15, 16]. This method is a discontinuous control strategy that involves selecting a switching surface for the desired dynamics and then designing a discontinuous control law. The system trajectory first reaches the surface and then stays on it as long as there are no perturbations [14].

In this paper, the goal is to synchronize two different chaotic systems: Lorenz system as master and Rössler system as slave. To achieve the goal, two robust controllers are designed: Active sliding mode and LQG/LTR controllers. First, the active sliding mode controller is designed as a combination of two design procedures: the active controller and the sliding mode controller [17]. The technique requires two stages. The first stage is to select an appropriate active controller to facilitate the design of the sequent sliding mode controller. The second stage is to design a sliding mode controller to achieve the synchronization. In this paper, the active sliding mode is implemented and in order to improve the method introduced in [17] uncertainty and disturbance are added to the slave system. Secondly, LQG/LTR controller is designed to achieve the required specifications which are described as good tracking and disturbance rejection by augmenting the open-loop system with integral action in each loop, robust stability, and robust performance of the closed-loop system with respect to the uncertainty of the plant. The LQG/LTR method is rooted in optimal control theory and is a systematic design approach based on shaping and recovering open-loop singular values [18]. LQG design approach is the combination of the optimal state-feedback regulator (LQR) and the Kalman filter (KF). Then, to recover the properties of both methods, the loop transfer recovery (LTR) is utilized. The LQG approach accompanied by LTR technique is named LQG/LTR method [18].

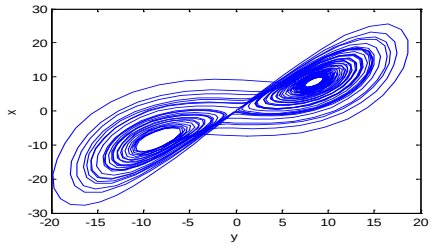
The organization of this paper is as follows: Section II focuses on the description of the dynamics of the Lorenz and Rössler chaotic systems, while the introduction of the active sliding mode method and the simulation resultants of the synchronization of two chaotic systems are provided in Section III. The design of the LQG/LTR method and the corresponding simulation results are described in Section IV. The comparison of the simulation results of two methods is given in Section V. Finally, conclusions are presented in Section VI.

II. THE DYNAMICS OF LORENZ AND RÖSSLER CHAOTIC SYSTEMS

The Lorenz system is known as a simplified model of several physical systems. The Lorenz equations may describe such different systems as laser devices, disk dynamos and several other problems related to the convection [19].The system is the first chaotic system which was introduced by Edward Lorenz in 1963. The Lorenz system is described by the following nonlinear differential equations:

$$\begin{cases} \dot{x} = \alpha(y - x) \\ \dot{y} = -xz + \gamma x - y \\ \dot{z} = xy - \beta z \end{cases} \quad (1)$$

where x, y and z are state variables and α, β and γ are three real constants. Figure 1 illustrates that the system has chaotic attractor when $\alpha = 10, \beta = 8/3$ and $\gamma = 28$.



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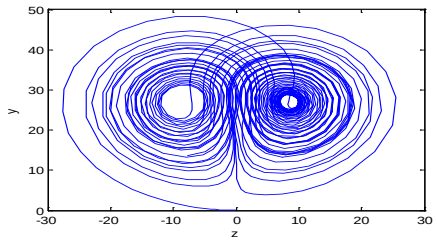

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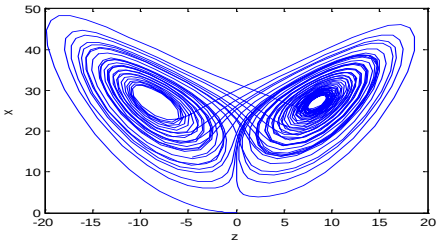

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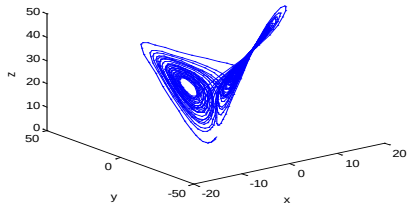

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Fig. 1 Lorenz chaotic attractor: a) projection on x-y, b) projection on y-z, c) projection on x-z, d) in x-y-z space

In the 1970s, Otto Rössler, a non-practicing medical doctor, became interested in chaos and concocted a three-dimensional autonomous system even simpler than Lorenz [20]. The dynamics of the Rössler system is described by:

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = -x + ay \\ \dot{z} = b + z(x - c) \end{cases} \quad (2)$$

where x, y and z are state variables and a, b and c are system parameters. Figure 2 shows the trajectories of the Rössler attractor where $a = b = 0.2$ and $c = 5.7$.

III. ACTIVE SLIDING MODE CONTROLLER DESIGN

The active sliding mode controller is a combination of two design procedures: the active controller and the sliding mode controller, which are given in succession.

A. Active Sliding Mode Controller Design

Consider the master chaotic system described by the following nonlinear differential equation:

$$\dot{x}(t) = f_1(x) \quad (3)$$

where $x(t) \in R^3$ denotes the system's three-dimensional state vector and $f_1 : R^3 \to R^3$ is the system's nonlinear equation. The slave system is described as below, in which $u(t) \in R^3$ as the control input and $M(t, y) \in R^3$ as the uncertainty and disturbance term are added to it:

$$\dot{y}(t) = f_2(y) + u(t) + M(t, y) \quad (4)$$

where $y(t) \in R^3$ is the slave system's three-dimensional state vector and $f_2 : R^3 \to R^3$ imply the similar role as f_1 for the master system. If $f_1 = f_2$, then x and y are the states of two identical chaotic systems. Otherwise, they represent the states of two different chaotic systems. We consider that:

$$M(t, y) = \Delta f(y) + d(t) \quad (5)$$

where $\Delta f \in R^3$ and $d(t) \in R^3$ are the uncertainty and the disturbance of the slave system, respectively. It is considered that the uncertainty and disturbance are bounded, such that:

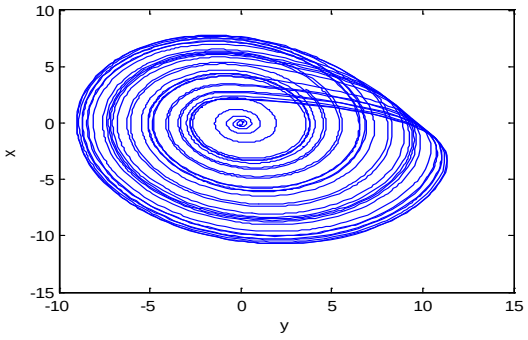
$$\|\Delta f\| < M_1 \quad (6)$$

$$\|d(t)\| < M_2 \quad (7)$$

where $M_1, M_2 \in R^3$ are the vectors of constant scalars. Then we can assume that:

$$0 < M_1 + M_2 < M_3 \quad (8)$$

where $M_3 \in R^3$ is a constant vector.



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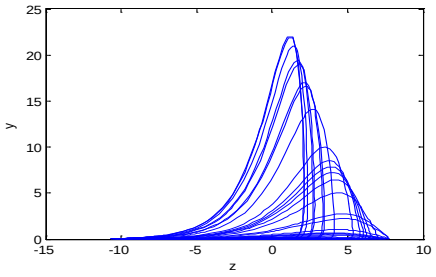

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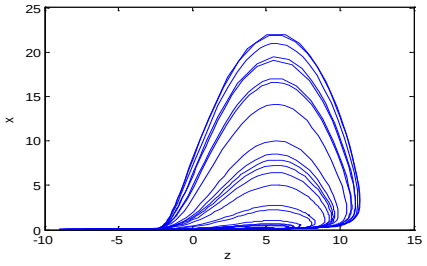

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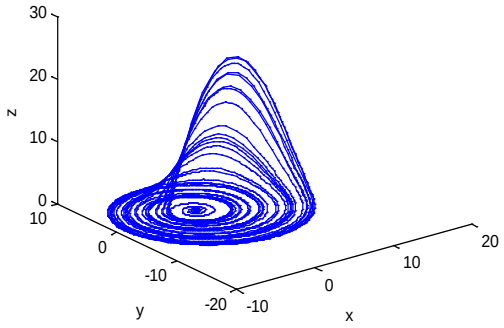

(d)

Fig. 2 Rössler chaotic attractor , a) projection on x-y, b) projection on y-z, c) projection on x-z, d) in x-y-z space

Therefore, we have:

$$\|M(t,y)\| < M_3. \tag{9}$$

As $M(t,y)$ represents the disturbance and uncertainty, which is unknown, but according to (9), it is bounded and its norm is less than M_3.

The synchronization problem is to design a controller which synchronizes the states of the master and slave systems. The dynamics of the synchronization error can be expressed as:

$$\dot{e} = f_2(y) - f_1(x) + u(t) + M(t,y) \tag{10}$$

where $e = y - x$.

The goal is to design the controller $u(t)$ such that:

$$\lim_{t\to\infty}\|e(t)\| = 0. \tag{11}$$

According to the active control design procedure, one uses the control input $u(t)$ to eliminate the nonlinear part of the error dynamics [17]. In other words, the input vector is considered as:

$$u(t) = Ae(t) + f_1(x) - f_2(y) + Kw(t) \tag{12}$$

where A is a canonical matrix and represents the linear part of the error dynamics, $K = [k_1, k_2, k_3]^T \in R^3$ is a constant gain vector and $w(t) \in R$ is the control input that satisfies:

$$w(t) = \begin{cases} w^+(t) & S(e) \geq 0 \\ w^-(t) & S(e) < 0 \end{cases} \tag{13}$$

and $S = S(e)$ is a switching surface which determines the desired dynamics.

The error system (10) is then rewritten as below:

$$\dot{e} = Ae(t) + Kw(t) + M(t,y). \tag{14}$$

In next section, the appropriate sliding mode controller will be designed according to the sliding mode control theory.

1) Sliding Surface Design

The sliding surface can be defined as follows:

$$S(e) = Ce \tag{15}$$

where $C^T = [c_1, c_2, c_3]^T \in R^3$ is a constant vector. The equivalent control approach is found by the fact that $\dot{S}(e) = 0$ is a necessary condition for the state trajectory to stay on the switching surface $S(e) = 0$. Hence, in the sliding mode, the controlled system satisfies the following conditions:

$$S(e) = 0 \tag{16}$$

and

$$\dot{S}(e) = 0. \tag{17}$$

Now, using (14), (15) and (17), one can obtain

$$\dot{S}(e) = \frac{\partial S(e)}{\partial e}\dot{e}(t) = C[Ae(t) + Kw(t) + M(t,y)] = 0. \tag{18}$$

Solving (18) for $w(t)$ results in the equivalent control $w_{eq}(t)$, as below:

$$w_{eq}(t) = -(CK)^{-1}(CAe). \tag{19}$$

Replacing $w_{eq}(t)$ for $w(t)$ in (14), the state equation in the sliding mode is as follows:

$$\dot{e}(t) = [I - K(CK)^{-1}C]Ae. \tag{20}$$

As long as the system (20) has all eigenvalues with negative real parts, it is asymptotically stable. For this purpose, the matrices K and C must be chosen properly.

2) Design of the Sliding Mode Controller

We assume that the constant plus proportional rate plus uncertainty norm reaching law is applied. The reaching law can be chosen as below:

$$\dot{S}(t) = -CK\ \mathrm{sgn}(S) - rS + CM_3 \tag{21}$$

where $\mathrm{sgn}(.)$ denotes the sign function and $M_3 \in R^3$ with positive constants is the additive norm of the uncertainty and disturbance term. The gain vectors C, K and the scalar r must be chosen such that the sliding condition is satisfied and the sliding mode motion occurred.

From (14) and (15), it can be found that

$$\dot{S}(t) = C[Ae(t) + Kw(t) + M(t,y)]. \tag{22}$$

Now, from (21) and (22), the control input is determined as below:

$$w(t) = -(CK)^{-1}[C(rI + A)e(t) + CK\ \mathrm{sgn}(S)]\cdot \tag{23}$$

B. Stability Analyse

The stability of the proposed method is studied in two parts: the convergence to the switching surface and the stability of the closed-loop system, which are given in the following.

1) Analyze of the Convergence to the Switching Surface

The error system in the sliding mode is determined as follows:

$$\begin{aligned}\dot{e}(t) = {} & [A - K(CK)^{-1}C(rI + A)]e(t) \\ & - K\ \mathrm{sgn}(S) + M(t,y)\end{aligned} \tag{24}$$

which is a three-dimensional linear system. The error dynamics can be reduced to a two-dimensional one, as:

$$e_3 = -\sum_{i=1}^{2} c_i e_i \tag{25}$$

where c_i is a positive scalar and $e_i = y_i - x_i$.

We suppose that

$$\left\| -K\ \mathrm{sgn}(S) + M(t,y) \right\| < \Phi \tag{26}$$

where $\Phi \in R^3$ is a constant vector with positive scalars.

Substituting (25) and (26) into the system (24), one can obtain:

$$\dot{E} = BE + \Phi \tag{27}$$

where $E = [e_1, e_2]^T \in R^2$ and $B \in R^{2\times 2}$. Using sub-matrix multiplication, the *i,j* th entry $b_{i,j}$ of matrix B can be calculated easily:

$$\begin{aligned} b_{i,j} = {} & a_{i,j} - a_{i,n}c_j - (CK)^{-1}K_i[Ca^j - c_j(Ca^n)] \\ & i, j = 1,2, \quad n = 2\end{aligned} \tag{28}$$

where $a_{i,j}$ is the i,j th entry of the matrix A and a^j denotes the j th column in A. All the eigenvalues of the system (27) has negative real parts. So, all the states of the system converge to the sliding surface.

2) Analyze of the Stability of the System

To check the stability of the controlled system, we consider the following Lyapunov candidate function:

$$V = \frac{1}{2}S^2. \tag{29}$$

The time derivative of (29) is as follows:

$$\dot{V} = S\dot{S} = S(C(Ae(t) + Kw(t) + M(t,y))). \tag{30}$$

By replacing (23) into (30), we obtain:

$$\dot{V} = -CKS\ \mathrm{sgn}(S) - rS^2 + SCM_3. \tag{31}$$

By choosing

$$CK > CM_3 - rS \tag{32}$$

we have $\dot{V} < 0$. The inequality (32) can be simplified to:

$$CK > CM_3\cdot \tag{33}$$

The matrices C and K must be chosen such that (33) holds. According to the Lyapunov theory, the above property implies the boundedness of the sliding surface S.

C. Simulation Results

Active sliding mode approach is applied to synchronize Lorenz and Rössler chaotic systems. The technique capability is examined by numerical simulations, which is discussed in the following.

To observe the synchronization behavior in the Lorenz and Rössler systems, we assume that the Lorenz system drives the Rössler system which contains uncertainty and disturbance. Therefore, the master and slave systems are defined, respectively, as follows:

$$\begin{cases} \dot{x}_1 = \alpha(x_2 - x_1) \\ \dot{x}_2 = -x_1x_3 + \gamma x_1 - x_2 \\ \dot{x}_3 = x_1x_2 - \beta x_3 \end{cases} \tag{34}$$

and

$$\begin{cases} \dot{y}_1 = -y_2 - y_3 + d_1(t) + \Delta f_1(y) + u_1(t) \\ \dot{y}_2 = y_1 + ay_2 + d_2(t) + \Delta f_2(y) + u_2(t) \\ \dot{y}_3 = b + y_3(y_1 - c) + d_3(t) + \Delta f_3(y) + u_3(t)\cdot \end{cases} \tag{35}$$

By subtracting (34) from (35), we obtain the error dynamics as below:

$$\begin{cases} \dot{e}_1 = -y_2 - y_3 + \alpha(x_2 - x_1) \\ \qquad + d_1(t) + \Delta f_1(y) + u_1(t) \\ \dot{e}_2 = y_1 + ay_2 + x_1x_3 - \gamma x_1 + x_2 \\ \qquad + d_2(t) + \Delta f_2(y) + u_2(t) \\ \dot{e}_3 = b + y_3(y_1 - c) + x_1x_2 + \beta x_3 \\ \qquad + d_3(t) + \Delta f_3(y) + u_3(t) \end{cases} \tag{36}$$

where $d_i(t)$ is external disturbance which is assumed to be a Gaussian white noise and Δf_i is uncertainty which is considered as below:

$$\begin{cases} \Delta f_1(y) = 0.5\prod_{i=1}^{3}\sin(i\pi y_i) \\ \Delta f_2(y) = 0.5\prod_{i=1}^{3}\sin(i\pi y_i) \\ \Delta f_3(y) = 0.5\prod_{i=1}^{3}\sin(i\pi y_i) \end{cases} \quad (37)$$

The control parameters are chosen as below:

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ K &= [0 \quad 1 \quad 1]^T \\ C &= [0.5 \quad 0.5 \quad 0.1] \\ r &= 20 \cdot \end{aligned} \quad (38)$$

As it is mentioned before, the matrices K and C are chosen such that the inequality (33) holds.

The above selections result in negative eigenvalues for the error dynamics. The control signal $w(t)$ is determined as follows:

$$w(t) = [-16.6667 \quad -17.5 \quad -4.16667]e(t) + \mathrm{sgn}(S(e)) \cdot \quad (39)$$

We selected the parameters of the Lorenz system as $\alpha = 10, \beta = 8/3, \gamma = 28$ and the parameters of the Rössler system as $a = b = 0.2$ and $c = 5.7$. The initial values of the master and slave systems are $x_1(0) = 0, x_2(0) = -0.01, x_3(0) = 9$ and $y_1(0) = -9, y_2(0) = 0$, $y_3(0) = 0$, respectively. These choices result in the initial error values of $e_1(0) = -9, e_2(0) = 0.01, e_3(0) = -9$.

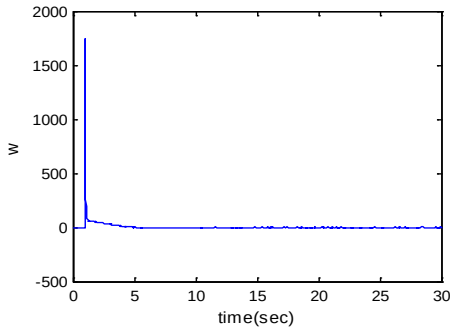

Fig. 3 Control signal $w(t)$ where controller is activated at t=1 sec

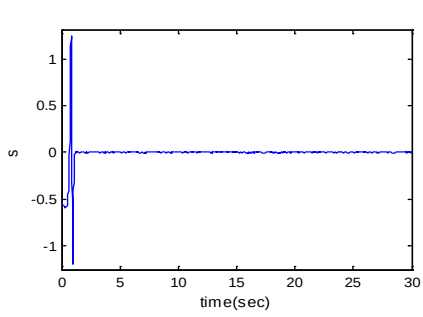

Fig. 4 Active sliding mode switching surface where controller is activated at t=1 sec

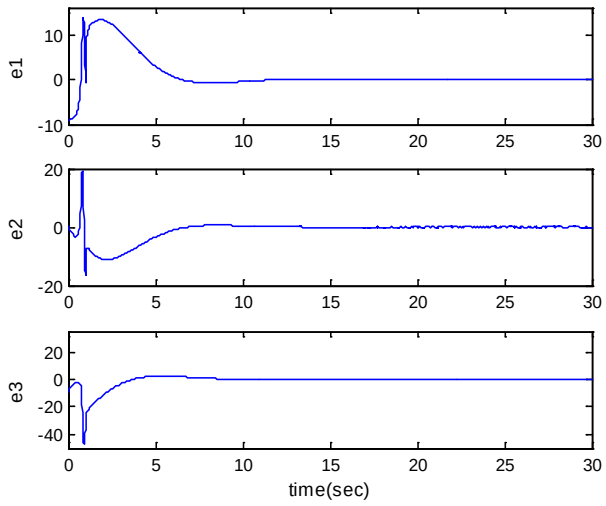

Fig. 5 Synchronization error signals where active sliding mode controller is activated at t=1 sec

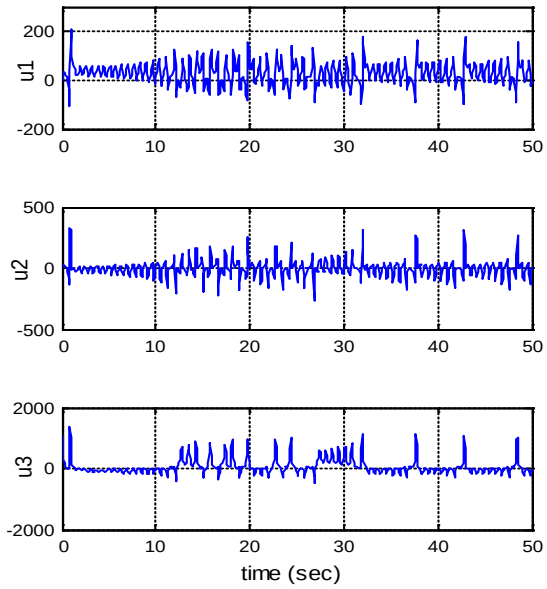

Fig. 6 Active sliding mode control signals applied at t=1 sec

The simulation results are given above. Figure 3 shows the control signal $w(t)$ which is applied at t=1 sec, while the switching surface is depicted in Figure 4. The error dynamics of the synchronization is illustrated in Figure 5. The control signals $u_i(t), \ i = 1,2,3$ which are applied to the Rössler system are shown in Figure 6.

IV. LQG/LTR DESIGN METHOD

A. *Problem Definition*

Suppose that we have a plant model in state space form, as follows:

$$\begin{cases} \dot{x} = Ax + Bu + \Gamma w \\ y = Cx + Du + v \end{cases} \quad (40)$$

where W and V are 'white noises', and

$$E\{ww^T\}=W>0$$
$$E\{vv^T\}=V>0 \quad (41)$$
$$E\{wv\}=0\cdot$$

The problem is to design a feedback control law, which minimizes the following cost function:

$$J=\lim_{t\to\infty}E\{\int_0^\infty (z^TQz+u^TRu)dt\} \quad (42)$$

where $z=Mx$ is some linear combination of the states and $Q=Q^T>0,\ R=R^T>0$ are the weighting matrices [18].

The solution to the LQG problem is presented by the separation principle, which states that the optimal result is achieved by the following procedure: first, one should obtain an optimal estimate $\hat{x}$ of the state x and then use this estimate as if it were an exact measurement of the state to solve a deterministic linear quadratic control problem. This procedure reduces the problem in two sub-problems, the solution to which is known [18].

B. The Solution of the LQG Problem

As mentioned above, the separation principle reduces the problem in two sub-problems. The first sub-problem is the state estimation. It is addressed by Kalman filter theory. The second sub-problem is to find the control signal which will minimize the following cost function:

$$\int_0^\infty (z^TQz+u^TRu)dt\,. \quad (43)$$

On the assumption that

$$\dot{x}=Ax+Bu+\Gamma w \quad (44)$$

the solution to this is to let the control signal u be the linear function of the states, as below:

$$u=-K_cx \quad (45)$$

where K_c is known as the state-feedback gain matrix [18, 21]. The optimal state-feedback matrix K_c is given by:

$$K_c=R^{-1}B^TP_c \quad (46)$$

where Pc satisfies the following algebraic Ricatti equation:

$$A^TP_c+P_cA-P_cBR^{-1}B^TP_c+M^TQM=0 \quad (47)$$

and $P_c=P_c^T\geq 0$. The Kalman filter gain matrix K_f is given, as below:

$$K_f=P_fC^TV^{-1} \quad (48)$$

where P_f satisfies the following algebraic Ricatti equation:

$$P_fA^T+AP_f-P_fC^TV^{-1}CV^{-1}P_f+\Gamma W\Gamma^T=0 \quad (49)$$

and $P_f=P_f^T\geq 0$ [18].

The resulting closed loop system is robust against any unstructured multiplicative uncertainty Δ, at the input of the plant, for which we have:

$$\bar{\sigma}(\Delta(jw))\leq\frac{1}{2}. \quad (50)$$

C. Loop Transfer Recovery (LTR)

LQG design approach is the combination of the optimal state-feedback regulator (LQR) and the Kalman filter (KF) as illustrated in Figure 7. Since both the LQR and KF methods have such good properties, it might be expected that LQG compensator would generally yield good robustness and performance. Unfortunately this is not the case: as Kalman filter loop gives good robustness and tracking properties, so we are in search of a compensator that recovers all these properties at the plant input. Similarly optimal state-feedback controller exhibits good robustness in the presence of unstructured uncertainties, therefore it is necessary for a Kalman filter to recover all these properties at the plant output. This procedure is called Loop Transfer Recovery. The former is called loop transfer recovery at the plant input and the later is called LTR at the plant output. At this stage, the dynamics of the plant are inversed and that is why LQG/LTR procedure is not valid for non square plant. During this, some of the filter eigenvalues are placed at the zeros of the plant, the remainder being allowed to be arbitrary fast.

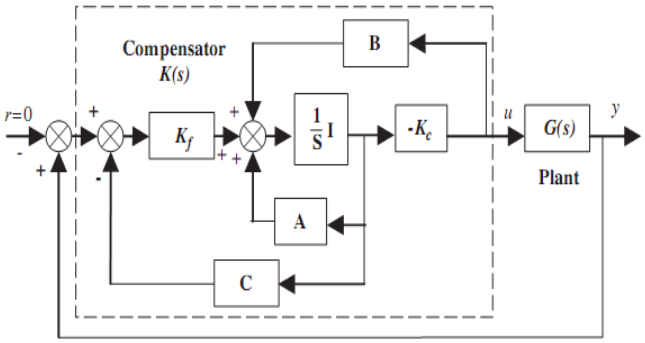

Fig. 7 Complete diagram of LQG/LTR method [9]

D. General Design Procedure

The following two step design approach is used for the LQG/LTR controller design for square plants.

Step-1: Design a Kalman filter by manipulating the covariance matrices W and V until the return ratio $-C(sI-A)^{-1}K_f$ obtained is satisfactory at the plant output. Emphasis should be put on the aspects such as 0-dB cross over frequencies for the principal gains, possibly balancing the principal gains [18].

Step-2: Design an optimal state-feedback by setting $M=C, Q=Q_0+qI$ and $R=\rho I$ (alternatively choose $W=I, V=I$). Increase q (or reduce ρ) until the return ratio of the compensated plant has converged sufficiently closely to $-C(jwI-A)^{-1}K_f$ over a sufficiently large range of frequencies [18].

It is often desired to design the return ratio at the plant output instead of the plant input. Otherwise, optimal state feedback matrix has to be designed in step-1 and then Kalman filter in step-2.

E. Simulation Results

Based on the procedure described in the previous sub-section, an LQG/LTR controller is designed for the synchronization of the Lorenz-Rössler chaotic systems. Loop transfer recovery is designed at the plant output, it means that, first the Kalman filter matrix (K_f) is calculated by setting $W = I,\ V = I$ and by shaping the principal gains in four steps as discussed in [18], then the optimal state feedback matrix (K_c) is obtained by setting $M = C,\ Q = I$ and $R = \rho I$. In the Kalman filter design procedure, we manipulate the singular values of the open-loop plant.

As the open-loop system is augmented with integral action in each loop, the closed-loop system has good tracking and disturbance attenuation according to simulation results. Figure 8 compares the principal gains of the return ratio of the original error system and the system after shaping the principal gains in four steps.

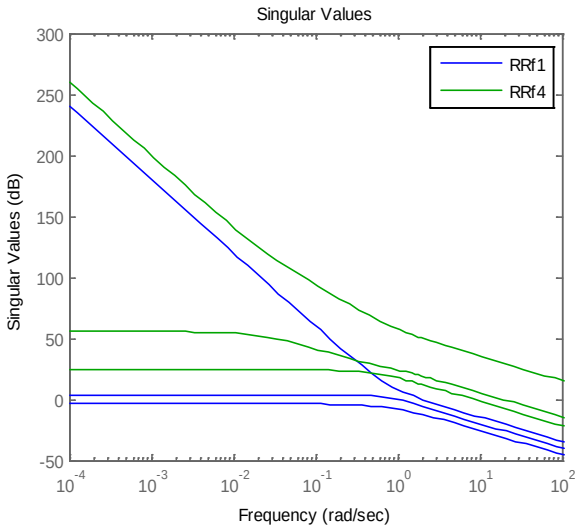

Fig. 8 Principal gains of the open-loop return ratio of the error dynamics; RRf1: original error system, RRf4: system after shaping the principal gains in four steps

The resulting observer gain matrix and optimal state feedback gain matrix are as below:

$$K_f = \begin{bmatrix} 12.0741 & -7.4608 & 0.2312 \\ -7.4608 & 12.9679 & 0.2631 \\ 0.2312 & 0.2631 & 4.4504 \\ 108.2113 & -101.9002 & 1.0998 \\ -98.0712 & 111.6867 & -2.2043 \\ 0.4950 & 0.6117 & 9.9645 \end{bmatrix} \tag{51}$$

$$K_c = \begin{bmatrix} 1000 & 0.5 & 0 & 1 & 0 & 0 \\ 0.5 & 1000 & 0.5 & 0 & 1 & 0 \\ 0 & 0.5 & 1000 & 0 & 0 & 1 \end{bmatrix}.$$

The designed controller in state space form $[A_c, B_c, C_c, D_c]$ is as follows:

$$\begin{aligned} A_c &= A_1 - B_1 K_c - K_f C_1 + K_f D_1 K_c \\ B_c &= K_f \\ C_c &= K_c \\ D_c &= 0 \end{aligned} \tag{52}$$

where $[A_1, B_1, C_1, D_1]$ is the augmented system model. The order of the designed controller is 6.

In this paper, LQG/LTR control method is implemented to synchronize two different chaotic systems. The designed controller is tested in MATLAB simulink, in which a Gaussian white noise is applied at the plant input.

The error signals, which are shown in Figure 9, confirm that Rössler system is synchronized with Lorenz system. Figure 10 depicts corresponding control signals which are applied at t=1 sec.

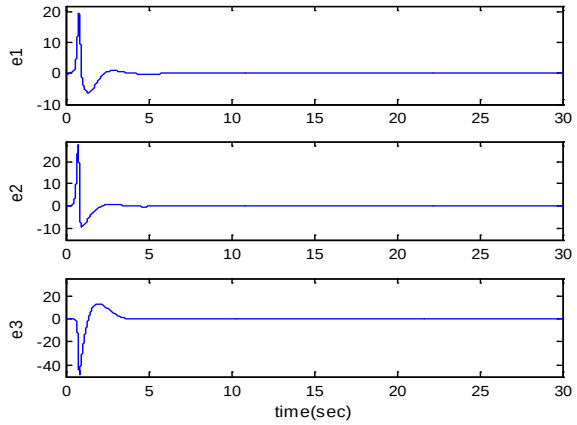

Fig. 9 Synchronization error signals where LQG/LTR controller is activated at *t*=1 sec

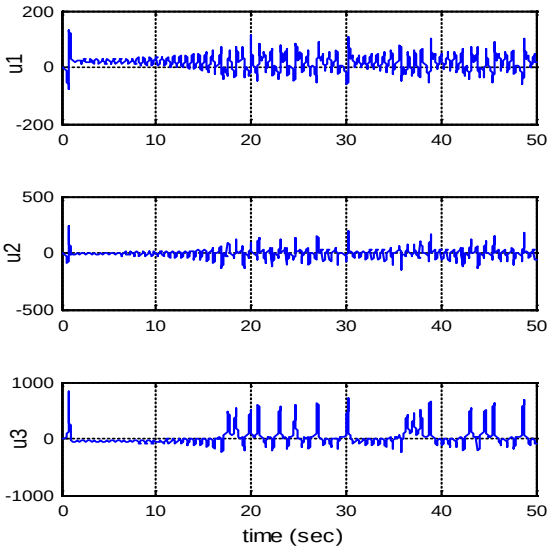

Fig. 10 LQG/LTR control signals applied at *t*=1 sec

V. THE COMPARISON OF THE RESULTS

In this paper, the goal was to synchronize two different chaotic systems: Lorenz system as master and Rössler system as slave. To achieve this, two robust controllers were designed: active sliding mode and LQG/LTR controllers. First, the active sliding mode controller was designed as a combination of two design procedures: the active controller and the sliding mode controller. In this method an external disturbance which was a Gaussian white noise and uncertainty term considered as sine function of the state variables were applied to the slave system. The simulation results showed that two systems are successfully synchronized. Secondly, LQG/LTR controller was designed which was a systematic design approach based on shaping and recovering open-loop singular values and provided stability automatically. The corresponding simulations illustrated that the synchronization errors converged to zero very fast while a Gaussian white noise was applied at the plant input.

As shown in Figure 11, the comparison of the error signals of two proposed methods demonstrates that the LQG/LTR control method provides better convergence rate to zero and less mean value of the error. According to Figure 11, the synchronization error signals e_1, e_2 and e_3 of the LQG/LTR control method reach zero when $t = 6.5\,\text{sec}$, $6\,\text{sec}$ and, $6\,\text{sec}$, respectively, while the error signals e_1, e_2 and e_3 of the active sliding mode method converge to zero when $t = 12.5\,\text{sec}$, $13\,\text{sec}$ and $10\,\text{sec}$, respectively. In addition, the comparison of the peak value of the error signals after applying control signals at $t = 1\,\text{sec}$ confirms the advantage of the LQG/LTR technique.

As shown in Figure 12, control efforts of both controllers for synchronizing two chaotic systems are in acceptable range, but the active sliding mode-based controller has chattering, as its inherent property, consequently LQG/LTR controller provides lower energy control signals. Besides, as both master and slave systems are chaotic, the control signals to synchronize the two systems have chaotic-like behaviour but they are bounded for all the times.

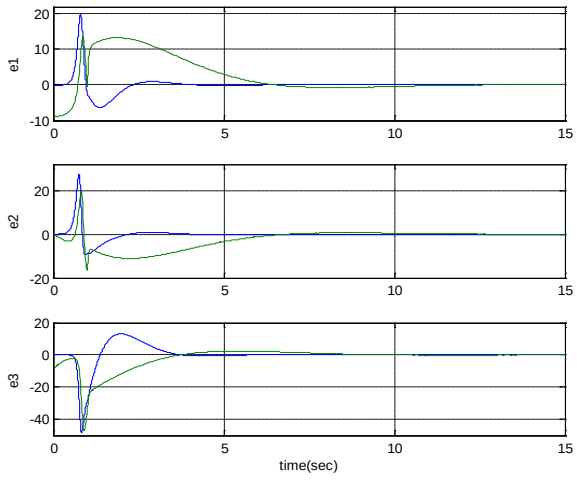

Fig. 11 The synchronization error signals of two designed controllers; (solid: LQG/LTR method, dashed: Active sliding mode)

VI. CONCLUSION

In this paper, two robust controllers were used for the synchronization of Lorenz–Rössler chaotic systems based on active sliding mode and LQG/LTR methods. The active control scheme had been successfully applied to synchronize both the response Rössler system and the drive Lorenz system, which provided efficient control law to make system robust with respect to system unmodeled dynamics and external disturbances.

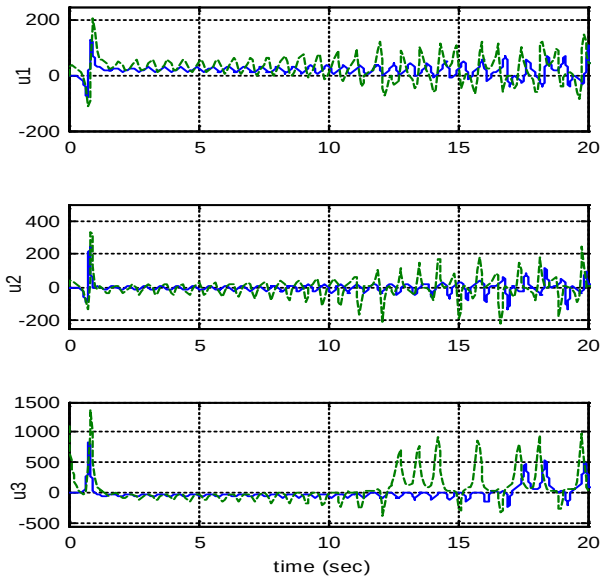

Fig. 12 The synchronization control signals of two designed controllers; (solid: LQG/LTR method, dashed: Active sliding mode)

The LQG/LTR controller was designed in two steps: first Kalman filter matrix was obtained by shaping the principal gains, and then the optimal state-feedback matrix was calculated. Due to the simulation results, the synchronization of Lorenz-Rössler systems was achieved successfully using this method. As LQG/LTR method provided better convergence rate to zero with less mean value of the error signals, and lower energy control signals, this technique has more advantages.

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